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An asymptotic theory of clad inhomogeneous planar waveguides: I. Eigenfunctions and the eigenvalue equation

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Abstract. Asymptotic representations are obtained for the eigenfunctions of the differential equation describing scalar waves in a clad inhomogeneous planar waveguide, which has two turning points and finite boundaries. These representations are valid to all asymptotic orders in the large parameter, which is proportional to wavenumber. This is achieved, in contrast to augmented WKB theory, by finding transformations on the independent variable which map the eigenfunction exactly into known solutions of canonical differential equations. The resulting transformation equations are nonlinear but have tractable asymptotic properties.

1. Introduction

The analysis of the propagation of light in a clad dielectric waveguide is currently an area of much activity. From the practical point of view, these guiding structures are of great importance in modern high-capacity communications systems, and from the theoretical point of view, they present features of interest not previously encountered, or of no fundamental significance, in more traditional forms of waveguides. Amongst these features, the most prominent results from the multimode nature of these guides and the attendant problem of pulse dispersion due to the different group velocity of each mode. A realistic assessment of this phenomenon requires, amongst other things, a high degree of accuracy in calculating the group velocities across the mode spectrum.

Originally, the WKB method was applied to this problem (Gloge and Marcatili 1973), with some success. Unfortunately it is known that WKB is inaccurate, or fails completely, in certain circumstances. In the optical waveguide context these occur when caustics interact with dielectric boundaries or with each other. In such cases alternative methods must be sought. In addition, if one wishes to remain within the area of asymptotic methods, an alternative theory must be capable of generating *several asymptotic orders* of approximation in order to fulfil the accuracy requirement referred to previously.

Recently, the present author considered the application of the uniform asymptotic theory of second-order differential equations with two turning points to a similar problem (Arnold 1980a, b), and equations were obtained from which the required parameters could be calculated in circumstances in which WKB fails. This work was based on the mathematical foundations laid down by Lynn and Keller (1970) and Olver (1975), along with a suitable ansatz to scale the required eigenvalue in such a manner as to render the calculations required more tractable.

In essence, the problem is to find the eigenvalue U^2 of the differential equation

$$d^2\phi/dx^2+(U^2-V^2f^2)\phi=0 \tag{1.1}$$

when $V\to\infty$, subject to the boundary condition

$$d\phi/dx=\pm K\phi \qquad x=\pm 1 \tag{1.2}$$

where K is some constant. (These equations are discussed in more detail in § 2). The function f^2 is assumed to be an even function of x, of polynomial form, such that $U^2-V^2f^2$ has only two zeros in $-1\leqslant x\leqslant 1$ for $U\leqslant V$. The restriction to symmetry of f is in no way essential, but it does simplify the analysis at certain points.

The standard uniform approximation procedure consists of applying the Liouville transform

$$\phi=(d\xi/dx)^{-1/2}\Phi \tag{1.3}$$

$$(\xi^2-\xi_0^2)(d\xi/dx)^2=f^2-f_0^2 \tag{1.4}$$

to equation (1.1), with

$$f_0^2=U^2/V^2. \tag{1.5}$$

This brings (1.1) to the form

$$d^2\Phi/d\xi^2+[V^2(\xi_0^2-\xi^2)+h]\Phi=0 \tag{1.6}$$

where h is an analytic function of ξ in a non-vanishing neighbourhood of the real ξ axis, excluding the points at infinity. In addition, h is $O(1)$ as $V\to\infty$, and is given explicitly as

$$h=\left(\frac{d\xi}{dx}\right)^{-3/2}\frac{d^2}{dx^2}\left(\frac{d\xi}{dx}\right)^{-1/2}. \tag{1.7}$$

These properties of h enable one to neglect it in (1.5) and to obtain

$$\Phi\sim\Phi_0 \tag{1.8}$$

where

$$d^2\Phi_0/d\xi^2+V^2(\xi_0^2-\xi^2)\Phi_0=0. \tag{1.9}$$

The parameter ξ_0 is chosen by the requirement that $\xi\to\xi_0$ as $x\to x_2$ and $\xi\to-\xi_0$ as $x\to x_1$, where x_1 and x_2 are the two zeros of $f^2-f_0^2$. Thus

$$\int_{-\xi_0}^{\xi_0}(\xi_0^2-\xi^2)^{1/2}\,d\xi=\int_{x_1}^{x_2}(f^2-f_0^2)^{1/2}\,dx. \tag{1.10}$$

Boundary conditions, obtained using (1.3), (1.4) and (1.8) in (1.2), are used to determine the correct solution of (1.9) (which is known to be a linear combination of Weber functions), and thus to obtain the parameter ξ_0. Equation (1.10) is then solved for f_0^2, and the eigenvalue U^2 follows trivially.

In practice, this method has a number of disadvantages. Firstly, the resulting expressions are dependent only implicitly on the original variables x, ϕ and f. For example, the exact solution to (1.4) cannot in general be obtained with ξ expressed in the form of an explicit function of x (except in the trivial case $f=x$, when $\xi=x$). Further approximation needs to be applied to extract explicit expressions for the eigenvalue.

Secondly, higher-order terms are more difficult to deal with; because of the implicit nature of the Liouville transform, and hence of subsequent higher-order terms, it is prohibitively laborious to extract *explicit* approximations to higher order. This is a serious disadvantage in applications, in view of the accuracy required.

Thirdly, the asymptotic expansion for ϕ obtained when higher-order terms are included is non-uniform as $|x| \to \infty$ (Olver 1975). Although we wish only to consider $|x| \leqslant 1$, this non-uniformity does have implications for the *accuracy* of asymptotic approximations.

The theory to be described in this paper has the objective of remedying these defects. In particular, explicit representations for ϕ will be obtained which can be continued to *all asymptotic orders* of approximation, regarding the eigenvalue simply as a parameter to be determined. This is achieved by obtaining pairs of linearly independent asymptotic solutions to (1.1) which are valid over more restricted regions than the full uniform solution. These can be obtained explicitly to all orders (though not in a general closed form). A uniform approximation is then introduced for the *specific purpose* of normalising these approximate solutions to a single exact solution ϕ (or, equivalently, finding connection formulae for the normalisation constants). These asymptotic solutions resemble WKB solutions, with Airy function forms to span turning points. However, there is an important difference between the expressions obtained here and the more traditional methods. Previous methods (Lynn and Keller 1970, Olver 1975) proceed by applying a definite transformation to the independent variable, x, and then seeking to approximate the dependent variable, ϕ. Here, however, the roles are reversed: we pose the problem of finding a transformation on x which will transform ϕ into a function Φ which is a *known solution* of a previously selected differential equation. In other words, what transformation, different from (1.4), will render $h \equiv 0$ in (1.5)? This idea appears to have been originated by Miller and Good (1953). Berry and Mount (1972) give several references, but the basic idea has not been systematically exploited as is described here, nor has it been applied to eigenvalue problems on *finite* domains. Problems of this type require the determination of linearly independent pairs of solutions near boundaries which must be correctly normalised to a single global function ϕ, a requirement which greatly increases the complexity of such problems over those to which these methods were originally applied (in quantum chemistry).

The resulting improvements in calculational effectiveness, best appreciated in actual application, are significant, removing all the defects of previous methods, and introducing a much greater flexibility to asymptotic methods. This flexibility is in part due to the explicit nature of the representations, extensible to all asymptotic orders, and partly due to the natural occurrence of *contour* integrals, in place of *definite* integrals in which the turning point is an integration limit.

2. Formulation of the problem

Suppose a planar waveguide is composed of an inhomogeneous medium of refractive index $n(x)$ for $-a \leqslant x \leqslant a$, surrounded by a homogeneous medium of refractive index $n = n_2$. Scalar waves in such a medium are composed of modes of the form $\phi(x)\,\mathrm{e}^{\mathrm{i}\beta z}$, where ϕ satisfies

$$\mathrm{d}^2\phi/\mathrm{d}x^2 + (n^2k^2 - \beta^2)\phi = 0 \tag{2.1}$$

where k is the free-space wavenumber, z is the direction of propagation, and it is assumed that there is no variation in the y direction. By letting

$$n^2 = n_0^2 - (n_0^2 - n_2^2)f^2 \tag{2.2}$$

$$V^2 = (n_0^2 - n_2^2)k^2a^2 \tag{2.3a}$$

$$U^2 = (n_0^2k^2 - \beta^2)a^2 \tag{2.3b}$$

where f is a function of x such that $f = 0$ at $x = 0$, we obtain

$$a^2\,\mathrm{d}^2\phi/\mathrm{d}x^2 + (U^2 - V^2f^2)\phi = 0. \tag{2.4}$$

Units of length can be chosen so that $a = 1$. Then

$$\mathrm{d}^2\phi/\mathrm{d}x^2 + (U^2 - V^2f^2)\phi = 0. \tag{2.5}$$

Equation (2.5) describes the field ϕ inside $-1 \leqslant x \leqslant 1$. Outside this region we have

$$\mathrm{d}^2\phi/\mathrm{d}x^2 - W^2\phi = 0 \tag{2.6}$$

where

$$W^2 = V^2 - U^2. \tag{2.7}$$

Equation (2.7) has the elementary solution (bounded as $|x| \to \infty$)

$$\phi = \pm A\,\mathrm{e}^{-W|x|} \tag{2.8}$$

where A is an arbitrary constant. Requiring that ϕ and $\mathrm{d}\phi/\mathrm{d}x$ be continuous at $x = \pm 1$ leads to the boundary condition

$$(1/\phi)\,\mathrm{d}\phi/\mathrm{d}x = \pm W \qquad x = \pm 1. \tag{2.9}$$

Thus the problem is to solve (2.5) on $-1 \leqslant x \leqslant 1$, subject to the boundary condition (2.9).

These equations describe propagation in a planar optical waveguide when $|(1/n^2)\,\mathrm{d}n^2/\mathrm{d}x| \ll 1$. They also describe propagation in acoustic ducts and many other situations of physical interest.

To make the refractive index n more definite we can assume that f is of polynomial form

$$f^2 = \sum_{j=0}^{M} a_j x^{2j+2} \qquad f^2(0) = 0 \qquad f^2(1) = 1. \tag{2.10}$$

where M is a finite integer. The $\{a_j\}$ are chosen such that they are real, and so that $U^2 - V^2f^2$ has only two zeros for $-1 \leqslant x \leqslant 1$, $U \leqslant V$. This is a realistic restriction in most cases of practical interest.

We will be particularly concerned with finding asymptotic approximations to ϕ and U^2 when the parameter V becomes large.

3. The transformations

Let us first consider the differential equation from a general point of view. Let

$$F^2 = f^2 - f_0^2 \tag{3.1a}$$

$$f_0^2 = U^2/V^2 \tag{3.1b}$$

so that the differential equation (1.1) becomes

$$\mathrm{d}^2\phi/\mathrm{d}x^2 - V^2F^2\phi = 0. \tag{3.2}$$

This differential equation is characterised asymptotically ($V \to \infty$) by points at which $F^2 = 0$; these are known as caustics, or turning points. They divide the interval of interest ($-1 \leqslant x \leqslant 1$) into disjoint regions where F^2 is alternately positive or negative. When $F^2 < 0$ solutions of (3.2) are oscillatory functions of x, and when $F^2 > 0$ they are smooth functions with exponential behaviour. Near caustics they must be described by transition functions.

With this classification scheme in mind, we shall describe a class of transformations of (3.2), representatives of which generate asymptotic approximations to ϕ to arbitrarily large order in the parameter $V^{-2}(V \to \infty)$. In particular, we seek a transformation $x \to t$, $\phi \to \Phi$ such that

$$\mathrm{d}^2\Phi/\mathrm{d}t^2 - V^2G^2\Phi = 0 \tag{3.3}$$

where G^2 is any convenient function of t for which (3.3) has a known solution. It is easily verified by substitution that such a transformation is

$$\phi = (\mathrm{d}t/\mathrm{d}x)^{-1/2}\Phi \tag{3.4a}$$

$$G^2\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^2 = F^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right)^{-1/2}. \tag{3.4b}$$

It is possible to regard such transformations as constituting a *group* (the group properties of closure, identity and inversion are easily established). The group property is important as it allows passage between different representations for ϕ, which is invariant under such transformations.

Representatives of the transformation group form subgroups when the independent variables of differential equations of the form (3.2)–(3.3) are restricted to intervals which contain a definite number of turning points, this number being invariant under the transformation, and a consideration of the analytical representations of the transformation in various subgroups leads to asymptotic approximations for ϕ, as we now show. In the following,

$$I = \{x : -1 \leqslant x \leqslant 1\} \qquad \text{and} \qquad I' \subset I$$

and the device will frequently be used of writing integrals of the form $\int^x h(x')\,\mathrm{d}x'$ as $\int^x h\,\mathrm{d}x'$ when h has previously been defined as a function of x (thus it is implicitly required that x be replaced by x' in the equations which define the integrand).

3.1. I' contains no turning points; $F^2 > 0$ for $x \in I'$

In this case, we may take $G^2 = 1$, since then (3.3) has a known solution

$$\Phi = A_1\,\mathrm{e}^{-Vt} + A_2\,\mathrm{e}^{Vt} \tag{3.5}$$

where A_1 and A_2 are arbitrary constants. It is convenient to choose a new variable, ζ say, and to reserve t for the general case; thus, with $\zeta = t$, and $G^2 = 1$, we have

$$\phi = (\mathrm{d}\zeta/\mathrm{d}x)^{-1/2}(A_1\,\mathrm{e}^{-V\zeta} + A_2\,\mathrm{e}^{V\zeta}) \tag{3.6a}$$

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^2 = F^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{-1/2}. \tag{3.6b}$$

Equation (3.6*b*) is a nonlinear differential equation for ζ. By letting

$$w = \mathrm{d}\zeta/\mathrm{d}x \tag{3.7}$$

it becomes

$$w^2 = F^2 - \frac{1}{V^2} w^{1/2} \frac{\mathrm{d}^2}{\mathrm{d}x^2} w^{-1/2} \tag{3.8a}$$

$$= F^2 + \frac{1}{V^2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4w^2}\frac{\mathrm{d}w^2}{\mathrm{d}x}\right) - \left(\frac{1}{4w^2}\frac{\mathrm{d}w^2}{\mathrm{d}x}\right)^2\right] \tag{3.8b}$$

which can be formally solved for $V \to \infty$ by iteration:

$$w^2 \sim F^2 + \frac{1}{V^2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right) - \left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)^2\right] + \mathrm{O}(V^{-4}). \tag{3.9}$$

Equation (3.9) is an asymptotic approximation to w^2; higher-order terms are generated by continuing the iteration, and the resulting series is uniform wherever these subsequent terms are vanishingly small as $V \to \infty$. The form of (3.9) suggests that the series is not uniform at points where $F^2 = 0$. At such points all higher-order terms are singular, being more highly singular the higher the order of the term.

The square root of (3.9) may be taken, to yield

$$w \sim F + \frac{1}{2V^2F}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right) - \left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)^2\right] + \mathrm{O}(V^{-4}). \tag{3.10}$$

Branches can be chosen for w and F such that they are positive when x is real, $x \in I'$, for sufficiently large V. In addition, $w^{-1/2}$ is to be positive under the same conditions.

It now remains to integrate w to obtain ζ, as prescribed by (3.7). It is essential for later analysis to fix the arbitrary constant appearing in this integration, and this is done by the following argument. Suppose (3.10) is analytically continued into the complex x-plane; then, traversing a closed contour around a zero of F^2 in the complex x-plane changes the sign of w. Hence, we may write

$$\zeta = \tfrac{1}{2}\int_\Gamma w \,\mathrm{d}x' \tag{3.11}$$

where Γ is a contour in the complex x'-plane starting at $x' = x$ on the *lower* Riemann sheet of w and passing around the zero of F^2, which is a branch point for w, to $x' = x$ on the *upper* Riemann sheet (x is replaced by x' as the integration variable) (Froman 1970). (See figure 1.)

The question arises as to whether w can be analytically continued along Γ, as it has previously been defined only on the real x-axis. This in turn raises some difficult mathematical questions concerning the existence of solutions to (3.8) and their uniqueness, and in what sense the asymptotic approximation (3.10) represents the true solution. It is not particularly helpful to go into these questions here. It turns out that solutions of (3.8) do exist, and the theory of (3.8) can be related to the theory of the generalised Riccati equation. The function described by (3.10) is uniformly asymptotic to a solution of (3.8) in the complex x-plane with a neighbourhood of each turning point deleted, subject to branch cuts which are introduced to make it single-valued. Thus, w in (3.11) can be replaced by its asymptotic expansion (3.10) analytically continued along Γ, provided that Γ does not pass through a zero of F^2.

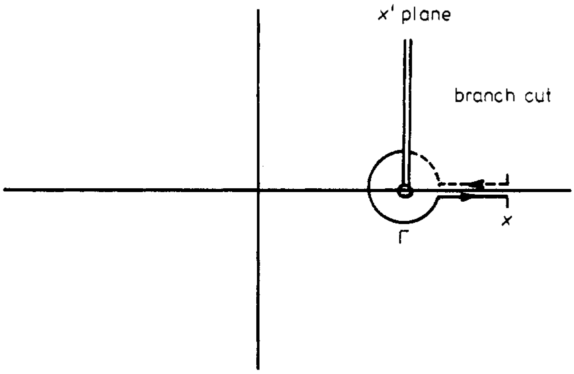

Figure 1. The contour Γ in the x' plane. Broken lines lie on the lower sheet of w.

Termwise integration of the expansion for w yields an expansion for ζ, which may be used in (3.6a) to approximate ϕ to arbitrarily large order.

The classical Liouville–Green (Olver 1974) or WKB (Froman and Froman 1965) approximations may be obtained by neglecting all higher-order terms in (3.10) which tend to zero as $V \to \infty$. Then

$$\zeta \sim \int_{x_0}^{x} (f^2 - f_0^2)^{1/2}\, dx' \tag{3.12}$$

and

$$\phi \sim (f^2 - f_0^2)^{-1/4}\left[A_1 \exp\left(-V\int_{x_0}^{x} (f^2 - f_0^2)^{1/2}\, dx'\right) + A_2 \exp\left(V\int_{x_0}^{x} (f^2 - f_0^2)^{1/2}\, dx'\right)\right] \tag{3.13}$$

where x_0 is the relevant zero of the integrand (the caustic).

3.2. I' contains no turning points; $F^2 < 0$ for $x \in I'$

The transformation in this case is represented by $t = \eta$, $G^2 = -1$. Then (3.3) has the elementary solution

$$\Phi = B_1\, e^{-iV\eta} + B_2\, e^{iV\eta} \tag{3.14}$$

and it is found that

$$\phi = (d\eta/dx)^{-1/2}(B_1\, e^{-iV\eta} + B_2\, e^{iV\eta}) \tag{3.15a}$$

$$\left(\frac{d\eta}{dx}\right)^2 = -F^2 + \frac{1}{V^2}\left(\frac{d\eta}{dx}\right)^{1/2} \frac{d^2}{dx^2}\left(\frac{d\eta}{dx}\right)^{-1/2} \tag{3.15b}$$

where B_1 and B_2 are constants. By letting

$$u = (d\eta/dx) \tag{3.16}$$

we obtain from (3.15b)

$$u^2 = -F^2 + \frac{1}{V^2} u^{1/2} \frac{d^2}{dx^2} u^{-1/2} \tag{3.17}$$

and the solution

$$u^2 = -w^2 \tag{3.18}$$

follows immediately, with w^2 given by (3.9). Hence

$$u \sim (-F^2)^{1/2} - \frac{(-F^2)^{-1/2}}{2V^2}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right) - \left(\frac{1}{4F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x}\right)^2\right] + \mathrm{O}(V^{-4}) \tag{3.19}$$

and

$$\eta = \tfrac{1}{2}\int_\Gamma u \,\mathrm{d}x'. \tag{3.20}$$

Branches for u and $(-F^2)^{1/2}$ can be chosen to make them positive for $x \in I'$, and similarly for $u^{-1/2}$.

3.3. *I' contains one turning point at $x = x_0$*

The canonical transformation here is given by $t = \tau$, $G^2 = \tau$. Then (3.3) becomes

$$\mathrm{d}^2\Phi/\mathrm{d}\tau^2 - V^2\tau\Phi = 0 \tag{3.21}$$

which has the solution

$$\Phi = C_1 \,\mathrm{Ai}(V^{2/3}\tau) + C_2 \,\mathrm{Bi}(V^{2/3}\tau)$$

where C_1 and C_2 are arbitrary constants, and $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$ are the standard Airy functions (Abramowitz and Stegun 1965). Equations (3.4a) and (3.4b) become

$$\phi = (\mathrm{d}\tau/\mathrm{d}x)^{-1/2}[C_1 \,\mathrm{Ai}(V^{2/3}\tau) + C_2 \,\mathrm{Bi}(V^{2/3}\tau)] \tag{3.22a}$$

$$\tau\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^2 = F^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{-1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{1/2}. \tag{3.22b}$$

To solve (3.22b), we introduce a useful device to be known as a *counterterm*. Let

$$\tau = \tau' + \tau_0'/V^2. \tag{3.23}$$

Then (3.22b) can be written as

$$\tau'\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^2 = F^2 - \frac{1}{V^2}\left[\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^{-1/2} + \tau_0'\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^2\right]. \tag{3.24}$$

The parameter τ_0' is to be chosen so that $\tau' \to 0$ as $F^2 \to 0$ ($x \to x_0$); this requires that

$$\lim_{x\to x_0}\left[\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^{-1/2}\left(\frac{\mathrm{d}\tau'}{\mathrm{d}x}\right)^{-2}\right] = -\tau_0'. \tag{3.25}$$

Equation (3.24) can now be solved explicitly by iteration; the first few terms are

$$\begin{aligned}\tfrac{2}{3}\tau'^{3/2} \sim \int_{x_0}^{x} F\,\mathrm{d}x' + \frac{1}{2V^2}\int_{x_0}^{x}\frac{1}{F}\Bigg[&\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x'}\left(\frac{1}{F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x'} - \frac{2}{3}\frac{F}{\int_{x_0}^{x'}F\,\mathrm{d}x''}\right)\\ &- \frac{1}{16}\left(\frac{1}{F^2}\frac{\mathrm{d}F^2}{\mathrm{d}x'} - \frac{2}{3}\frac{F}{\int_{x_0}^{x'}F\,\mathrm{d}x''}\right)^2 - \tau_0'^{(0)}F^2\left(\tfrac{3}{2}\int_{x_0}^{x'}F\,\mathrm{d}x''\right)^{-2/3}\Bigg]\mathrm{d}x' + \mathrm{O}(V^{-4}).\end{aligned} \tag{3.26}$$

The constant $\tau_0'^{(0)}$ is the leading-order term in τ_0', and is given by

$$\tau_0'^{(0)} = -\tfrac{3}{10}F_1^{-1/3}F_2 \tag{3.27}$$

where

$$F_1 = \mathrm{d}F^2/\mathrm{d}x|_{x=xo} \tag{3.28a}$$

$$F_2 = \mathrm{d}^2F^2/\mathrm{d}x^2|_{x=xo}. \tag{3.28b}$$

The conventional Langer approximation is obtained by neglecting all terms in (3.26) which vanish as $V\to\infty$.

3.4. I' contains two turning points $x = x_1$, $x = x_2$, $x_2 > x_1$

The canonical transformation in this case is given by $t = \xi$, $G^2 = \xi^2 - \xi_0^2$, and ξ_0^2 is some constant to be chosen to make ξ and x analytic functions of each other near caustics. Then we have

$$\mathrm{d}^2\Phi/\mathrm{d}\xi^2 - V^2(\xi^2 - \xi_0^2)\Phi = 0 \tag{3.29}$$

$$\phi = (\mathrm{d}\xi/\mathrm{d}x)^{-1/2}\Phi \tag{3.30a}$$

$$(\xi^2 - \xi_0^2)\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 = F^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1/2}. \tag{3.30b}$$

Equation (3.30*b*) cannot be solved explicitly except in a special case, which allows $\xi_0^2\to 0$. Suppose

$$f_0^2 = \chi/V \tag{3.31a}$$

$$\xi_0^2 = \chi'/V \tag{3.31b}$$

where χ and χ' are $\mathrm{O}(1)$ as $V\to\infty$. Then $f_0^2\to 0$ as $V\to\infty$, and the zeros of F^2 tend to the zeros of f^2, which in practice consist of a double zero at $x = 0$. Thus the caustics tend to coalesce at the origin; $x_1, x_2\to 0$.

Under these circumstances the terms containing ξ_0^2 and f_0^2 in (3.30*b*) are treated as counterterms, to give

$$\xi^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 = f^2 - \frac{1}{V}\left[\chi - \chi'\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2\right] - \frac{1}{V^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1/2}. \tag{3.32}$$

χ' is then chosen to make the sum of the last two terms on the right of (3.32) vanish as $x\to 0$, thus ensuring that ξ is an analytic function of x near the origin. Hence

$$\lim_{x\to 0}\left[\chi - \chi'\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^2 - \frac{1}{V}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1/2}\right] = 0. \tag{3.32b}$$

Then (3.30*b*) can be solved explicitly by iteration, with the result

$$\frac{\xi^2}{2} \sim \int_0^x f\,\mathrm{d}x' - \frac{1}{2V}\int_0^x\left(\frac{\chi}{f} - \frac{\chi'}{2}\frac{f}{\int_0^{x'} f\,\mathrm{d}x''}\right)\mathrm{d}x' \tag{3.33}$$

and

$$\chi' \simeq \chi\lim_{x\to 0}\left(\frac{2\int_0^x f\,\mathrm{d}x'}{f^2}\right) + \mathrm{O}(V^{-1}). \tag{3.34}$$

Higher-order terms follow without difficulty.

In general, an explicit solution of (3.30*b*) is difficult to obtain. However, the only use to be made of this fully uniform representation for ϕ is to normalise the representations given in §§ 3.1–3.3 above, and it turns out that for this purpose an explicit solution is not required, but merely the assumption of analyticity of the functions $x(\xi)$ and $\xi(x)$; ξ_0 is supposed to have been chosen to ensure this.

Solutions of (3.29) are known, and can be represented in the form (Buchholz 1961, Slater 1960)

$$\Phi^{\mathrm{e}} = \cos(\tfrac{1}{2}\nu\pi)\Phi_1^{\mathrm{e}} - \sin(\tfrac{1}{2}\nu\pi)\Phi_2^{\mathrm{e}} \tag{3.35a}$$

$$\Phi^{\mathrm{o}} = \sin(\tfrac{1}{2}\nu\pi)\Phi_1^{\mathrm{o}} + \cos(\tfrac{1}{2}\nu\pi)\Phi_2^{\mathrm{o}} \tag{3.35b}$$

where the superscripts refer to even and odd solutions, and

$$\Phi_1^{\mathrm{e}} = \frac{1}{2\pi\mathrm{i}}\int_{\infty\mathrm{e}^{-\mathrm{i}\sigma}}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} H^{\mathrm{e}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s \tag{3.36a}$$

$$\Phi_2^{\mathrm{e}} = \frac{1}{2\pi}\int_{-1}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} H^{\mathrm{e}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s + \frac{1}{2\pi}\int_{-1}^{\infty\mathrm{e}^{-\mathrm{i}\sigma}} H^{\mathrm{e}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s \tag{3.36b}$$

$$\Phi_1^{\mathrm{o}} = \frac{1}{2\pi\mathrm{i}}\int_{\infty\mathrm{e}^{-\mathrm{i}\sigma}}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} \xi H^{\mathrm{o}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s \tag{3.36c}$$

$$\Phi_2^{\mathrm{o}} = \frac{1}{2\pi}\int_{-1}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} \xi H^{\mathrm{o}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s + \frac{1}{2\pi}\int_{-1}^{\infty\mathrm{e}^{-\mathrm{i}\sigma}} \xi H^{\mathrm{o}}(s)\,\mathrm{e}^{-sZ/2}\,\mathrm{d}s \tag{3.36d}$$

with

$$H^{\mathrm{e}}(s) = (1-s)^{-\nu/2-1}(1+s)^{\nu/2-1/2} \tag{3.37a}$$

$$H^{\mathrm{o}}(s) = (1-s)^{-\nu/2-1/2}(1+s)^{\nu/2} \tag{3.37b}$$

$$V\xi^2 = Z \tag{3.38a}$$

$$V\xi_0^2 = 2(\nu+\tfrac{1}{2}) \tag{3.38b}$$

and $\sigma > 0$. The various fractional powers of the integrands are defined to be positive on the real axis between $s=-1$ and $s=1$. The two solutions (3.35) are linearly independent, as are the pairs $(\Phi_1^{\mathrm{e}}, \Phi_2^{\mathrm{e}})$ and $(\Phi_1^{\mathrm{o}}, \Phi_2^{\mathrm{o}})$. The representations are valid as long as the integrals converge, which covers $\xi\to\infty$ but not $\xi\to-\infty$; for negative real ξ the symmetry properties are used to calculate Φ^{e} and Φ^{o}.

Using the functions Φ^{e} and Φ^{o}, the most general representation for ϕ is

$$\phi = \gamma^{\mathrm{e}}\phi^{\mathrm{e}} + \gamma^{\mathrm{o}}\phi^{\mathrm{o}} \tag{3.39}$$

$$\phi^{\mathrm{e}} = (\mathrm{d}\xi/\mathrm{d}x)^{-1/2}\Phi^{\mathrm{e}} \qquad \phi^{\mathrm{o}} = (\mathrm{d}\xi/\mathrm{d}x)^{-1/2}\Phi^{\mathrm{o}} \tag{3.40}$$

where γ^{e} and γ^{o} are constants.

4. Normalisation and connection formulae

The central problem in this asymptotic theory is the normalisation of the various pairs of solutions to a single solution ϕ, or, in other words, the evaluation of the constants A_1, A_2, B_1, B_2, C_1 and C_2 in § 3. Here the group properties referred to at the beginning of § 3 are of fundamental significance. The basic principle involved will be illustrated for the case of the constants A_1 and A_2 of § 3.1.

In § 3, it was shown that ϕ has a representation

$$\phi=(\mathrm{d}\zeta/\mathrm{d}x)^{-1/2}(A_1\,\mathrm{e}^{-V\zeta}+A_2\,\mathrm{e}^{V\zeta}) \tag{4.1a}$$

where

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^2=F^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{-1/2}. \tag{4.1b}$$

This representation applies in any region for which $F^2>0$. For definiteness, we assume this to be the regio $x>x_2$, where x_1 and x_2 $(x_2>x_1)$ are the real zeros of F^2. Also in § 3.3 we considered a representation for ϕ in terms of a solution Φ of (3.29). Now (3.29) is a differential equation of the same type as (3.2); therefore Φ has a representation

$$\Phi=(\mathrm{d}\zeta^*/\mathrm{d}\xi)^{-1/2}(A_1^*\,\mathrm{e}^{-V\zeta^*}+A_2^*\,\mathrm{e}^{V\zeta^*}) \tag{4.2a}$$

where

$$\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\xi}\right)^2=G^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\xi}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}\xi}\right)^{1/2} \tag{4.2b}$$

$$G^2=\xi^2-\xi_0^2 \tag{4.3}$$

in the region $\xi>\xi_0$. By substituting the known relations (3.30a) and (3.30b) in (4.2), we obtain

$$\phi=(\mathrm{d}\zeta^*/\mathrm{d}x)^{-1/2}(A_1^*\,\mathrm{e}^{-V\zeta^*}+A_2^*\,\mathrm{e}^{V\zeta^*}) \tag{4.4a}$$

and

$$\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}x}\right)^2=F^2-\frac{1}{V^2}\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\mathrm{d}\zeta^*}{\mathrm{d}x}\right)^{-1/2}. \tag{4.4b}$$

Comparing (4.4) with (4.1), we observe that $\mathrm{d}\zeta/\mathrm{d}x$ and $\mathrm{d}\zeta^*/\mathrm{d}x$ satisfy the same equation. If conditions could be found which guarantee that $\zeta=\zeta^*$, then, comparing (4.4a) with (4.1a), it would also be necessary for

$$A_1=A_1^* \tag{4.5a}$$

$$A_2=A_2^*. \tag{4.5b}$$

Since A_1^* and A_2^* can be found from integral representations (3.36) for the condition that Φ must be single-valued, this would determine A_1 and A_2 also for the same condition on ϕ, because ξ and x are analytic functions of each other.

It can be shown that necessary and sufficient conditions for

$$\zeta=\zeta^* \tag{4.6}$$

are that

$$\int_{C_{x'}}\frac{\mathrm{d}\zeta}{\mathrm{d}x'}\,\mathrm{d}x'=\int_{C_{x'}}\frac{\mathrm{d}\zeta^*}{\mathrm{d}x'}\,\mathrm{d}x' \tag{4.7}$$

where $C_{x'}$ is *any* closed contour passing through $x'=x$. In addition, if x and ξ are connected by the transformation (3.30b) which is analytic in certain regions of the x and ξ planes, then

$$\int_{C_{x'}}\frac{\mathrm{d}\zeta^*}{\mathrm{d}x'}\,\mathrm{d}x'=\int_{C_{\xi'}}\frac{\mathrm{d}\zeta^*}{\mathrm{d}\xi}\,\mathrm{d}\xi' \tag{4.8}$$

where $C_{\xi'}$ is the image of $C_{x'}$ under the mapping $x' \to \xi'$ ($x \to \xi$ with x and ξ replaced by x' and ξ' for the purposes of integration) whenever $C_{x'}$ and $C_{\xi'}$ lie entirely within the respective domains of analyticity of the mapping. Hence (4.9*a*) and (4.9*b*) also imply that

$$\int_{C_{x'}} \frac{d\zeta}{dx'}\,dx' = \int_{C_{\xi'}} \frac{d\zeta^*}{d\xi'}\,d\xi'. \tag{4.9}$$

That (4.7) is necessary is not difficult to see; sufficiency requires a lengthy analysis of the possible forms that solutions of (4.1*b*) can take, and will not be pursued here.

Now ζ and ζ^* are defined by

$$\zeta = \frac{1}{2}\int_{\Gamma_{x'}} \frac{d\zeta}{dx'}\,dx' \tag{4.10a}$$

$$\zeta^* = \frac{1}{2}\int_{\Gamma_{\xi'}} \frac{d\zeta^*}{d\xi'}\,d\xi' \tag{4.10b}$$

where $\Gamma_{x'}$ and $\Gamma_{\xi'}$ are closed contours surrounding the respective branch points of $d\zeta/dx'$ and $d\zeta'/d\xi'$, passing through $x = x'$, $\xi = \xi'$ respectively. These contours clearly comply with the above conditions, and certainly $\zeta = \zeta^*$.

A further quantity of importance can be deduced by allowing $C_{x'}$ to surround both turning points; we call such a contour Γ_0. The corresponding integral on the right-hand side of (4.9) can be evaluated *exactly* by iteration of (4.2*b*) to any order, and substituting the result in (4.9). It turns out that to *all orders*

$$\frac{1}{2\pi i}\oint_{\Gamma_{\xi'}} \frac{d\zeta^*}{d\xi'}\,d\xi' = -\frac{\xi_0^2}{2} \tag{4.11}$$

where $\Gamma_{\xi'}$ is a closed contour surrounding the points $\xi' = \pm\xi_0$. Multiplying by $-V$, using (3.38*b*) and applying (4.9) results in

$$-\frac{V}{2\pi i}\oint_{\Gamma_0} \frac{d\zeta}{dx'}\,dx' = \nu + \tfrac{1}{2}. \tag{4.12}$$

Since the right-hand side of (4.12) is constant, Γ_0 may be any closed contour passing around the two branch points of the integrand and need not pass through x. The integration variable can therefore be changed to x, $d\zeta/dx$ replaced by w according to (3.7), and (4.12) written

$$\frac{-V}{2\pi i}\oint_{\Gamma_0} w\,dx = \nu + \tfrac{1}{2}. \tag{4.13}$$

An identical analysis of the function η defined in § 3.2 leads to the equation

$$\frac{-V}{2\pi}\oint_{\Gamma_0} u\,dx = \nu + \tfrac{1}{2}. \tag{4.14}$$

Equations (4.13) and (4.14) may be regarded as global conditions connecting x and ξ to each other to ensure that ξ is an analytic function of x within and on Γ_0. They are not independent because $w^2 = -u^2$. These equations will be important in obtaining the eigenvalue equation in § 5.

Having established that $\zeta = \zeta^*$, it follows that $A_1 = A_1^*$, $A_2 = A_2^*$, and it remains to find A_1^* and A_2^* from the integral representations for the confluent hypergeometric functions. This is achieved by allowing $\xi \to \infty$ along the real axis, estimating the functions Φ_1^e, Φ_2^e, Φ_1^o and Φ_2^o from (3.36) in the asymptotic ($\xi \to \infty$) limit, and comparing with the same limit estimated from (4.3) with $\zeta = \zeta^*$. This calculation is outlined in the following.

4.1. I' contains no turning points; $F^2 > 0$ for $x \in I'$

In addition to the above conditions we assume $x > x_2$.

The limiting forms of the functions defined in (3.36) are obtained by Laplace's method for $\xi \to \infty$. The results are

$$\Phi_1^e \sim D_1^e \xi^{\nu} \exp(-\tfrac{1}{2} V \xi^2) \tag{4.15a}$$

$$\Phi_2^e \sim D_2^e \xi^{-\nu-1} \exp(\tfrac{1}{2} V \xi^2) \tag{4.15b}$$

$$\Phi_1^o \sim D_1^o \xi^{\nu} \exp(-\tfrac{1}{2} V \xi^2) \tag{4.15c}$$

$$\Phi_2^o \sim D_2^o \xi^{-\nu-1} \exp(\tfrac{1}{2} V \xi^2) \tag{4.15d}$$

where

$$D_1^e = 2^{-1/2} V^{\nu/2} / \Gamma(\nu/2+1) \tag{4.16a}$$

$$D_2^e = (2^{-1/2}/\pi) V^{-\nu/2-1/2} \Gamma(\nu/2+\tfrac{1}{2}) \tag{4.16b}$$

$$D_1^o = 2^{1/2} V^{\nu/2-1/2} / \Gamma(\nu/2+\tfrac{1}{2}) \tag{4.16c}$$

$$D_2^o = (2^{1/2}/\pi) V^{-\nu/2-1} \Gamma(\nu/2+1). \tag{4.16d}$$

On the other hand, solving (4.2b) approximately (with $\zeta = \zeta^*$) as for (3.6b) leads to

$$(\mathrm{d}\zeta/\mathrm{d}\xi)^2 \sim \xi^2 - \xi_0^2 - (1/V^2)[\tfrac{3}{4}(\xi^2 - \xi_0^2)^{-1} + \tfrac{5}{4}\xi_0^2(\xi^2 - \xi_0^2)^{-2}] + \mathrm{O}(V^{-4}). \tag{4.17}$$

The last equation suggests that

$$(\mathrm{d}\zeta/\mathrm{d}\xi)^2 \sim \xi^2 - \xi_0^2 + \mathrm{O}(\xi^{-2}) \tag{4.18}$$

as $\xi \to \infty$, and this can be verified by conducting the iterative solution of (4.2b) while regarding ξ, rather than V, as a large parameter. Taking the square root of (4.18), integrating over a contour Γ as required by (4.11b), expanding the result for large ξ, and neglecting terms which vanish as $\xi \to \infty$ leads to the conclusion that

$$\zeta \sim \zeta_0 + \tfrac{1}{2}\xi^2 - \tfrac{1}{4}\xi_0^2 - \tfrac{1}{2}\xi_0^2 \ln(2\xi/\xi_0) \tag{4.19a}$$

as $\xi \to \infty$, where ζ_0 is a constant whose asymptotic expansion is

$$\zeta_0 \sim 1/24 V^2 \xi_0^2 + \mathrm{O}(V^{-4}). \tag{4.19b}$$

Multiplying (4.19a) and (4.19b) by V and using (3.38b) gives

$$V\zeta \sim V\zeta_0 + \tfrac{1}{2} V\xi^2 - \tfrac{1}{2}(\nu + \tfrac{1}{2}) - \ln(2\xi/\xi_0)^{\nu+1/2} \tag{4.19c}$$

and

$$V\zeta_0 \sim \tfrac{1}{48}(\nu + \tfrac{1}{2})^{-1} + \mathrm{O}(V^{-3}). \tag{4.19d}$$

We now express the various functions in (4.15) in the form

$$\Phi_1^e \sim A_1^{e'} (\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2} \mathrm{e}^{-V\zeta} \tag{4.20a}$$

$$\Phi_2^e \sim A_2^{e'} (\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2} \mathrm{e}^{V\zeta} \tag{4.20b}$$

$$\Phi_1^{\rm o} \sim A_1^{\rm o'}(\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2}\,\mathrm{e}^{-V\zeta} \tag{4.20c}$$

$$\Phi_2^{\rm o} \sim A_2^{\rm o'}(\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2}\,\mathrm{e}^{V\zeta} \tag{4.20d}$$

which is possible because $\Phi_1^{\rm e}$, $\Phi_1^{\rm o}$ are recessive solutions of (3.29) as $\xi \to \infty$, and $\Phi_2^{\rm e}$, $\Phi_2^{\rm o}$ are dominant solutions. (Strictly, (4.20*b*) and (4.20*d*) should have a recessive component added to them, but such a term would be negligible as $\xi \to \infty$, and its normalisation could not be determined.) Inserting the limiting form for ζ from (4.19) in (4.20) and comparing with (4.15) leads to the following result:

$$A_1^{\rm e'} = \frac{2^{-1/2}V^{\nu/2}\exp[-\frac{1}{2}(\nu+\frac{1}{2})]\exp(-V\zeta_0)}{\Gamma(\nu/2+1)}\left(\frac{\xi_0}{2}\right)^{\nu+1/2} \tag{4.21a}$$

$$A_2^{\rm e'}/A_1^{\rm e'} = (2\pi)^{-1/2}\Gamma(\nu+1)(\nu+\tfrac{1}{2})^{-\nu-1/2}\exp(\nu+\tfrac{1}{2})\exp(2V\zeta_0) \tag{4.21b}$$

$$A_1^{\rm o'} = \frac{2^{1/2}V^{\nu/2-1/2}\exp[-\frac{1}{2}(\nu+\frac{1}{2})]\exp(-V\zeta_0)}{\Gamma(\nu/2+\frac{1}{2})}\left(\frac{\xi_0}{2}\right)^{\nu+1/2} \tag{4.21c}$$

$$A_2^{\rm o'}/A_1^{\rm o'} = A_2^{\rm e'}/A_1^{\rm e'}. \tag{4.21d}$$

The particular form for the above equations is chosen because it is the ratios (4.21*b*) and (4.21*d*) which appear in the subsequent eigenvalue equation. We note here that if $\xi_0 \sim \mathrm{O}(1)$ as $V \to \infty$, then $\nu \sim \mathrm{O}(V)$ and the function $\Gamma(\nu+1)$ in (4.21) has the expansion

$$\Gamma(\nu+1) \sim (2\pi)^{1/2}(\nu+\tfrac{1}{2})^{\nu+1/2}\exp[-(\nu+\tfrac{1}{2})]\exp[-(\nu+\tfrac{1}{2})^{-1}/24V] \tag{4.22}$$

and using (4.22) and (4.19*d*) in (4.21*b*) suggests that

$$A_2^{\rm e'}/A_1^{\rm e'} = A_2^{\rm o'}/A_1^{\rm o'} = 1 \tag{4.23}$$

to all orders. This conjecture will be proved later.

4.2. *I′ contains no turning points; $F^2 < 0$, $x \in I'$*

We suppose the closure of the open interval I' to be the points x_1 and x_2, the two caustics. Now, the function η in (3.20) can be defined by passing the contour Γ around *either* turning point. This gives rise to two possible definitions for ϕ for $x_1 < x < x_2$:

$$\phi = (\mathrm{d}\eta/\mathrm{d}x)^{-1/2}[B_{11}\exp(-\mathrm{i}V\eta_1) + B_{21}\exp(\mathrm{i}V\eta_1)] \tag{4.24a}$$

or

$$\phi = (\mathrm{d}\eta/\mathrm{d}x)^{-1/2}[B_{12}\exp(\mathrm{i}V\eta_2) + B_{22}\exp(-\mathrm{i}V\eta_2)] \tag{4.24b}$$

where

$$\eta_1 = \tfrac{1}{2}\int_{\Gamma_1} u\,\mathrm{d}x' \tag{4.25a}$$

$$\eta_2 = \tfrac{1}{2}\int_{\Gamma_2} u\,\mathrm{d}x' \tag{4.25b}$$

$$\mathrm{d}\eta/\mathrm{d}x = u \tag{4.26}$$

and Γ_1 and Γ_2 pass from x on the lower sheet to x on the upper sheet around x_1 and x_2 respectively (see figure 2). Because (4.24*a*) and (4.24*b*) both represent ϕ on the same interval, and η_1 and η_2 differ only by an integration constant, a linear relation exists

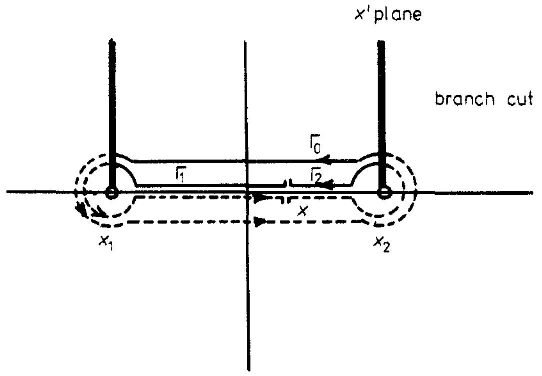

Figure 2. The contours Γ_0, Γ_1, Γ_2 in the x' plane. Broken lines lie on the lower sheet of u.

between the constants $\{B_{11}, B_{21}\}$ and $\{B_{12}, B_{22}\}$. Let

$$\eta_2 - \eta_1 = \tfrac{1}{2}\oint_{\Gamma_0} u \, \mathrm{d}x = \alpha \tag{4.27}$$

with Γ_0 surrounding both turning points. It can be verified that, with the choice of principal value for u as in § 3.2,

$$\eta_1 > 0 \tag{4.28a}$$

$$\eta_2 < 0 \tag{4.28b}$$

and therefore

$$\alpha < 0. \tag{4.29}$$

Substituting (4.27) in (4.24b) and equating the results to (4.24a) leads to the equations

$$B_{21} = B_{12}\,\mathrm{e}^{\mathrm{i}V\alpha} \tag{4.30a}$$

$$B_{11} = B_{22}\,\mathrm{e}^{-\mathrm{i}V\alpha} \tag{4.30b}$$

which are the required connection formulae.

By exactly identical reasoning to that used in § 4.1 above, it can be shown that Φ^{e} and Φ^{o} have similar representations:

$$\Phi^{\mathrm{e}} = (\mathrm{d}\eta/\mathrm{d}\xi)^{-1/2}[B_{11}^{\mathrm{e}} \exp(-\mathrm{i}V\eta_1) + B_{21}^{\mathrm{e}} \exp(\mathrm{i}V\eta_1)] \tag{4.31a}$$

$$= (\mathrm{d}\eta/\mathrm{d}\xi)^{-1/2}[B_{12}^{\mathrm{e}} \exp(\mathrm{i}V\eta_2) + B_{22}^{\mathrm{e}} \exp(-\mathrm{i}V\eta_2)] \tag{4.31b}$$

and

$$\Phi^{\mathrm{o}} = (\mathrm{d}\eta/\mathrm{d}\xi)^{-1/2}[B_{11}^{\mathrm{o}} \exp(-\mathrm{i}V\eta_1) + B_{21}^{\mathrm{o}} \exp(\mathrm{i}V\eta_1)] \tag{4.31c}$$

$$= (\mathrm{d}\eta/\mathrm{d}\xi)^{-1/2}[B_{12}^{\mathrm{o}} \exp(\mathrm{i}V\eta_2) + B_{22}^{\mathrm{o}} \exp(-\mathrm{i}V\eta_2)] \tag{4.31d}$$

where

$$\eta_1 = \frac{1}{2}\int_{\Gamma_1^*} \frac{\mathrm{d}\eta}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \tag{4.32a}$$

$$\eta_2 = \frac{1}{2}\int_{\Gamma_2^*} \frac{\mathrm{d}\eta}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \tag{4.32b}$$

and Γ_1^* and Γ_2^* are the *images* of Γ_1 and Γ_2 in (4.25) under the mapping $x \to \xi$; $\mathrm{d}\eta/\mathrm{d}\xi$ is a solution of

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^2 = G^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{-1/2} \tag{4.33a}$$

$$G^2 = \xi^2 - \xi_0^2. \tag{4.33b}$$

The asymptotic solution of (4.33*b*) is obtained by iteration:

$$(\mathrm{d}\eta/\mathrm{d}\xi)^2 \sim \xi_0^2 - \xi^2 - (1/V^2)[\tfrac{3}{4}(\xi_0^2-\xi^2)^{-1} - \tfrac{5}{4}(\xi_0^2-\xi^2)^{-2}] + \mathrm{O}(V^{-4}). \tag{4.33c}$$

Also, the relation (4.9) has the form

$$\oint_{\Gamma_0} \frac{\mathrm{d}\eta}{\mathrm{d}x'}\,\mathrm{d}x' = \oint_{\Gamma_{\xi'}} \frac{\mathrm{d}\eta}{\mathrm{d}\xi'}\,\mathrm{d}\xi' \tag{4.34}$$

where Γ_0 and $\Gamma_{\xi'}$ surround the turning points (branch points) of their respective integrands. For the right-hand side

$$\oint_{\Gamma_{\xi'}} \frac{\mathrm{d}\eta}{\mathrm{d}\xi'}\,\mathrm{d}\xi' = -\xi_0^2\pi \tag{4.35}$$

as can be proved by direct calculation. Using (4.35), (3.38*b*) and (3.16), (4.34) may be written as

$$-\frac{V}{2\pi}\oint_{\Gamma_0} u\,\mathrm{d}x = \nu + \tfrac{1}{2}. \tag{4.36}$$

Since $u^2 = -w^2$, this equation is completely consistent with (4.13).

Furthermore, relations analogous to (4.30) exist:

$$B_{21}^{\mathrm{e}} = B_{12}^{\mathrm{e}}\,\mathrm{e}^{-\mathrm{i}\nu\pi}\,\mathrm{e}^{-\mathrm{i}\pi/2} \tag{4.37a}$$

$$B_{11}^{\mathrm{e}} = B_{22}^{\mathrm{e}}\,\mathrm{e}^{\mathrm{i}\nu\pi}\,\mathrm{e}^{\mathrm{i}\pi/2} \tag{4.37b}$$

$$B_{21}^{\mathrm{o}} = B_{12}^{\mathrm{o}}\,\mathrm{e}^{-\mathrm{i}\nu\pi}\,\mathrm{e}^{-\mathrm{i}\pi/2} \tag{4.37c}$$

$$B_{11}^{\mathrm{o}} = B_{22}^{\mathrm{o}}\,\mathrm{e}^{\mathrm{i}\nu\pi}\,\mathrm{e}^{\mathrm{i}\pi/2}. \tag{4.37d}$$

Now, because of the symmetry of the confluent hypergeometric functions

$$B_{11}^{\mathrm{e}} = B_{12}^{\mathrm{e}} = B_1^{\mathrm{e}} \tag{4.38a}$$

$$B_{21}^{\mathrm{e}} = B_{22}^{\mathrm{e}} = B_2^{\mathrm{e}} \tag{3.38b}$$

$$-B_{11}^{\mathrm{o}} = B_{12}^{\mathrm{o}} = B_1^{\mathrm{o}} \tag{4.38c}$$

$$-B_{21}^{\mathrm{o}} = B_{22}^{\mathrm{o}} = B_2^{\mathrm{o}}. \tag{4.38d}$$

In addition, the confluent hypergeometric functions are real when ξ is real; therefore

$$B_2^{\mathrm{e}} = \bar{B}_1^{\mathrm{e}} \tag{4.39a}$$

$$B_2^{\mathrm{o}} = \bar{B}_1^{\mathrm{o}} \tag{4.39b}$$

where the bar means complex conjugation.

Using (4.38) and (4.39) in (4.37) leads to

$$B_1^{\mathrm{e}} = B^{\mathrm{e}}\, \mathrm{e}^{\mathrm{i}\nu\pi/2}\, \mathrm{e}^{\mathrm{i}\pi/4} \tag{4.40a}$$

$$B_2^{\mathrm{e}} = B^{\mathrm{e}}\, \mathrm{e}^{-\mathrm{i}\nu\pi/2}\, \mathrm{e}^{-\mathrm{i}\pi/4} \tag{4.40b}$$

$$B_1^{\mathrm{o}} = \mathrm{i}B^{\mathrm{o}}\, \mathrm{e}^{\mathrm{i}\nu\pi/2}\, \mathrm{e}^{\mathrm{i}\pi/4} \tag{4.40c}$$

$$B_2^{\mathrm{o}} = -\mathrm{i}B^{\mathrm{o}}\, \mathrm{e}^{-\mathrm{i}\nu\pi/2}\, \mathrm{e}^{-\mathrm{i}\pi/4} \tag{4.40d}$$

where B^{e} and B^{o} are real constants yet to be determined. With (4.40) in (4.31) we obtain

$$\Phi^{\mathrm{e}} = 2B^{\mathrm{e}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{-1/2}\left[\cos\left(\frac{\pi}{4} - V\eta_1\right)\cos\left(\frac{\nu\pi}{2}\right) - \sin\left(\frac{\pi}{4} - V\eta_1\right)\sin\left(\frac{\nu\pi}{2}\right)\right] \tag{4.41a}$$

$$= 2B^{\mathrm{e}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{-1/2}\left[\cos\left(\frac{\pi}{4} + V\eta_2\right)\cos\left(\frac{\nu\pi}{2}\right) - \sin\left(\frac{\pi}{4} + V\eta_2\right)\sin\left(\frac{\nu\pi}{2}\right)\right] \tag{4.41b}$$

$$\Phi^{\mathrm{o}} = 2B^{\mathrm{o}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{-1/2}\left[\cos\left(\frac{\pi}{4} - V\eta_1\right)\sin\left(\frac{\nu\pi}{2}\right) + \sin\left(\frac{\pi}{4} - V\eta_1\right)\cos\left(\frac{\nu\pi}{2}\right)\right] \tag{4.41c}$$

$$= 2B^{\mathrm{o}}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^{-1/2}\left[\cos\left(\frac{\pi}{4} + V\eta_2\right)\sin\left(\frac{\nu\pi}{2}\right) + \sin\left(\frac{\pi}{4} + V\eta_2\right)\cos\left(\frac{\nu\pi}{2}\right)\right] \tag{4.41d}$$

when referred to either turning point. The remaining constants B^{e} and B^{o} can be obtained by comparing the limiting values of Φ^{e} and Φ^{o} as $\xi \to 0$ from (4.41) with the corresponding values from the known integral or series representations for these functions. This calculation is straightforward, but tedious, and will only be outlined.

By the symmetry properties of the confluent hypergeometric functions, and (4.27) with (4.36),

$$\lim_{\xi\to 0} \eta_1 = -\lim_{\xi\to 0} \eta_2 = \tfrac{1}{2}(\nu + \tfrac{1}{2})\pi/V. \tag{4.42}$$

Therefore, from (4.41),

$$\lim_{\xi\to 0} \Phi^{\mathrm{e}} = 2B^{\mathrm{e}}c \tag{4.43a}$$

$$\lim_{\xi\to 0} \xi^{-1}\Phi^{\mathrm{o}} = 2B^{\mathrm{o}}c^{-1}V \tag{4.43b}$$

where

$$c = \lim_{\xi\to 0} (\mathrm{d}\eta/\mathrm{d}\xi)^{-1/2}. \tag{4.44a}$$

The constant c has an asymptotic expansion obtained directly from (4.33c):

$$c \sim \xi_0^{-1/2}[1 - 1/16V^2\xi_0^2 + \mathrm{O}(V^{-4})] \tag{4.44b}$$

which, using (3.38b), is

$$c \sim \xi_0^{-1/2}[1 - \tfrac{1}{32}(\nu + \tfrac{1}{2})^{-2} + \mathrm{O}(V^{-4})]. \tag{4.44c}$$

Comparing (4.43) with the corresponding estimates from the integral representations (3.36), we find

$$B^{\mathrm{e}} = \tfrac{1}{2}(2\pi/c)^{-1/2}[\Gamma(\nu/2 + \tfrac{1}{2})/\Gamma(\nu/2 + 1)] \tag{4.45a}$$

$$B^{\mathrm{o}} = 2cV^{-1}(2\pi)^{-1/2}[\Gamma(\nu/2 + 1)/\Gamma(\nu/2 + \tfrac{1}{2})]. \tag{4.45b}$$

4.3. I' contains one turning point

The normalisation of the Airy function representations of (3.22) follows similar principles to those used previously. There exists a representation for Φ in the form

$$\Phi = (\mathrm{d}\tau^*/\mathrm{d}\xi)^{-1/2}[C_1^* \operatorname{Ai}(V^{2/3}\tau^*) + C_2^* \operatorname{Bi}(V^{2/3}\tau^*)] \tag{4.46a}$$

$$\tau^*\left(\frac{\mathrm{d}\tau^*}{\mathrm{d}\xi}\right)^2 = G^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\tau^*}{\mathrm{d}\xi}\right)^{+1/2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\mathrm{d}\tau^*}{\mathrm{d}\xi}\right)^{-1/2}. \tag{4.46b}$$

If τ and τ^* are analytic functions of x and ξ respectively in neighbourhoods of the respective turning points of F^2 and G^2, then

$$\tau = \tau^* \tag{4.47a}$$

$$C_1 = C_1^* \qquad C_2 = C_2^* \tag{4.47b}$$

and (4.46) becomes

$$\Phi = (\mathrm{d}\tau/\mathrm{d}\xi)^{-1/2}[C_1 \operatorname{Ai}(V^{2/3}\tau) + C_2 \operatorname{Bi}(V^{2/3}\tau)] \tag{4.48a}$$

$$\tau\left(\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\right)^2 = G^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\mathrm{d}\tau}{\mathrm{d}\xi}\right)^{-1/2}. \tag{4.48b}$$

Coefficients C_1 and C_2 which ensure that Φ, and hence ϕ, is single-valued are obtained from the integral representations (3.36) for the single-valued functions Φ^{e} and Φ^{o}. The method used for this calculation requires the asymptotic estimation of τ as $\xi \to \infty$ along the Stokes' lines (Olver 1974), and this again uses the group properties of § 2.

Since the Airy functions satisfy equation (3.21), which is of the same type as (3.2), there must exist relations

$$\operatorname{Ai}(V^{2/3}\tau) = E_1(\mathrm{d}\zeta/\mathrm{d}\tau)^{-1/2}\,\mathrm{e}^{-V\zeta} \tag{4.49a}$$

$$\operatorname{Bi}(V^{2/3}\tau) = E_2(\mathrm{d}\zeta/\mathrm{d}\tau)^{-1/2}\,\mathrm{e}^{V\zeta} + E_3(\mathrm{d}\zeta/\mathrm{d}\tau)^{-1/2}\,\mathrm{e}^{-V\zeta} \tag{4.49b}$$

which take account of the recessive and dominant character respectively of these functions. The relation between τ and ζ is

$$\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^2 = \tau - \frac{1}{V^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}\tau}\right)^{-1/2}. \tag{4.50}$$

These relations are general properties of the Airy functions unrelated to any particular interpretations of ζ and τ. However, when x, ζ, ξ and τ are connected by the group of transformations we have considered, then (4.50) is a natural relation *within the group.*

The constants E_1 and E_2 are found by letting $\tau \to \infty$; then, from (4.50) we obtain

$$(\mathrm{d}\zeta/\mathrm{d}\tau)^2 \sim \tau + \mathrm{O}(\tau^{-2}) \tag{4.51}$$

i.e.

$$\tau \sim (\tfrac{3}{2}\zeta)^{2/3}. \tag{4.52}$$

Using this estimate and the standard asymptotic forms of the Airy functions, the constants E_1 and E_2 are found:

$$E_1 = \tfrac{1}{2}\pi^{-1/2}V^{-1/6} \tag{4.53a}$$

$$E_2 = \pi^{-1/2}V^{-1/6}. \tag{4.53b}$$

The constant E_3 is indeterminate, as it multiplies an exponentially small term which disappears in the limit $V \to \infty$; this is a characteristic of all asymptotic methods applied to this problem where exponentially increasing and decreasing functions appear together.

Now suppose ξ goes to infinity along some curve such that

$$\operatorname{Im}\left(\int_{\xi_0}^{\xi} (\xi'^2 - \xi_0^2)^{1/2}\, \mathrm{d}\xi'\right) = 0. \tag{4.54}$$

There are three such lines (Stokes' lines) emanating from each of the points $\xi = \pm\xi_0$. Along any line only one of the functions

$$\mathrm{Ai}(V^{2/3}\tau) \qquad \mathrm{Ai}(\mathrm{e}^{2\pi\mathrm{i}/3} V^{2/3}\tau) \qquad \mathrm{Ai}(\mathrm{e}^{-2\pi\mathrm{i}/3} V^{2/3}\tau)$$

is recessive. Along the same line, only one of the functions

$$\int_{\infty\mathrm{e}^{-\mathrm{i}\sigma}}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} H(s)\, \mathrm{e}^{-sZ/2}\, \mathrm{d}s \qquad \int_{-1}^{\infty\mathrm{e}^{\mathrm{i}\sigma}} H(s)\, \mathrm{e}^{-sZ/2}\, \mathrm{d}s \qquad \int_{-1}^{\infty\mathrm{e}^{-\mathrm{i}\sigma}} H(s)\, \mathrm{e}^{-sZ/2}\, \mathrm{d}s$$

is recessive. (Here the notation is the same as that of (3.36), with $H(s)$ in place of $H^{\mathrm{e}}(s)$ or $H^{\mathrm{o}}(s)$, as required.) Hence it is possible to pair off functions from each set, leaving only a multiplying constant to determine in each case. These constants are obtained by estimating the integrals for Φ_1^{e}, Φ_2^{e}, etc along the appropriate Stokes' line, and comparing with their Airy function representation using (4.52), (4.49a) and the standard asymptotic expansions of the Airy functions. For instance, we have already established that, for the Stokes' line formed by the real axis, $\xi > \xi_0$,

$$\Phi_1^{\mathrm{e}} = A_1^{\mathrm{e}'} (\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2}\, \mathrm{e}^{-V\zeta} \tag{4.55}$$

where $A_1^{\mathrm{e}'}$ is given by (4.21a); on the other hand, there is a representation

$$\Phi_1^{\mathrm{e}} = C_1^{\mathrm{e}'} (\mathrm{d}\tau/\mathrm{d}\xi)^{-1/2}\, \mathrm{Ai}(V^{2/3}\tau) \tag{4.56}$$

by matching the recessive representation to Φ_1^{e}. Using (4.51), (4.49a) and (4.53a), (4.56) becomes, as $\xi \to \infty$,

$$\Phi_1^{\mathrm{e}} \to \tfrac{1}{2} C_1^{\mathrm{e}'} V^{-1/6} \pi^{-1/2} (\mathrm{d}\zeta/\mathrm{d}\xi)^{-1/2}\, \mathrm{e}^{-V\zeta}. \tag{4.57}$$

Comparing (4.55) with (4.57) yields the constant $C_1^{\mathrm{e}'}$ in terms of $A_1^{\mathrm{e}'}$, which is known. In this way, we find if

$$\Phi^{\mathrm{e}} = (\mathrm{d}\tau/\mathrm{d}x)^{-1/2} [C_1^{\mathrm{e}}\, \mathrm{Ai}(V^{2/3}\tau) + C_2^{\mathrm{e}}\, \mathrm{Bi}(V^{2/3}\tau)] \tag{4.58a}$$

$$\Phi^{\mathrm{o}} = (\mathrm{d}\tau/\mathrm{d}x)^{-1/2} [C_1^{\mathrm{o}}\, \mathrm{Ai}(V^{2/3}\tau) + C_2^{\mathrm{o}}\, \mathrm{Bi}(V^{2/3}\tau)] \tag{4.58b}$$

that

$$C_1^{\mathrm{e}} = C_1^{\mathrm{e}'} \cos(\nu\pi/2) \tag{4.59a}$$

$$C_2^{\mathrm{e}} = -C_2^{\mathrm{e}'} \sin(\nu\pi/2) \tag{4.59b}$$

$$C_1^{\mathrm{o}} = C_1^{\mathrm{o}'} \sin(\nu\pi/2) \tag{4.59c}$$

$$C_2^{\mathrm{o}} = C_2^{\mathrm{o}'} \cos(\nu\pi/2) \tag{4.59d}$$

and

$$C_1^{\mathrm{e}'} = 2\pi^{1/2} V^{1/6} A_1^{\mathrm{e}'} \tag{4.60a}$$

$$C_2^{\mathrm{e}'} = 2\pi^{1/2} V^{1/6} A_2^{\mathrm{e}'} \tag{4.60b}$$

$$C_1^{o'} = 2\pi^{1/2} V^{1/6} A_1^{o'} \tag{4.60c}$$

$$C_2^{o'} = 2\pi^{1/2} V^{1/6} A_2^{o'}. \tag{4.60d}$$

Since these coefficients transfer directly to the representation for ϕ, we find that the most general representation for ϕ in the vicinity of a turning point is

$$\phi = \gamma^e \phi^e + \gamma^o \phi^o \tag{4.61a}$$

$$\phi^e = (\mathrm{d}\tau/\mathrm{d}x)^{-1/2}[C_1^e \, \mathrm{Ai}(V^{2/3}\tau) + C_2^e \, \mathrm{Bi}(V^{2/3}\tau)] \tag{4.61b}$$

$$\phi^o = (\mathrm{d}\tau/\mathrm{d}x)^{-1/2}[C_1^o \, \mathrm{Ai}(V^{2/3}\tau) + C_2^o \, \mathrm{Bi}(V^{2/3}\tau)] \tag{4.61c}$$

where γ^e and γ^o are arbitrary constants.

By an identical method to that used to derive (4.49) and the associated coefficients it can be shown that

$$\mathrm{Ai}(V^{2/3}\tau) = \pi^{-1/2} V^{-1/6} (\mathrm{d}\eta/\mathrm{d}\tau)^{-1/2} \cos(\pi/4 - V\eta) \tag{4.62a}$$

$$\mathrm{Bi}(V^{2/3}\tau) = \pi^{-1/2} V^{-1/6} (\mathrm{d}\eta/\mathrm{d}\tau)^{-1/2} \sin(\pi/4 - V\eta) \tag{4.62b}$$

when $\tau < 0$, where

$$\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2 = -\tau + \frac{1}{V^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^{1/2} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^{-1/2} \tag{4.63}$$

$$\eta = \frac{1}{2}\int_\Gamma \frac{\mathrm{d}\eta}{\mathrm{d}\tau'} \mathrm{d}\tau' \tag{4.64}$$

(note here that τ' is simply a replacement for τ for integration purposes, and is not the same as τ' in (3.23)) and Γ is a contour surrounding $\tau' = 0$ in the positive direction passing through $\tau' = \tau$.

The principal value of $\mathrm{d}\eta/\mathrm{d}\tau$ is positive for real τ, $\tau < 0$.

Using (4.62) and the normalisation (4.58)–(4.60) for the functions Φ^e and Φ^o, the normalisation for the oscillatory representation (the constants B_1^e, B_2^e, B_1^o and B_2^o) can be recalculated; it then turns out that

$$A_1^{e'} = B^e \qquad A_1^{o'} = B^o \tag{4.65a}$$

$$A_2^{e'}/A_1^{e'} = 1 \qquad A_2^{o'}/A_1^{o'} = 1. \tag{4.65b}$$

Although these relations are not obvious by inspection of the relevant equations (4.21) and (4.45), they are verifiable by direct calculation asymptotically to finite orders. This is an extremely tedious process beyond the leading order.

5. Quantisation

Having obtained normalised representations for the (unique) single-valued solution ϕ, it remains to apply the boundary conditions and so determine the eigenvalues of the differential equation. This calculation is conducted by a straightforward procedure, using different representations for ϕ for different configurations of caustics and boundaries. It is convenient to assume that the profile function f is symmetric ($f(-x) = -f(x)$) as this simplifies the analysis, but it is not essential to restrict the problem in this way. For symmetric profiles, ϕ^e and ϕ^o form *independent* eigenfunctions. This means that the pair (A_1, A_2) in (4.1) can be taken to be either

(A_1^e, A_2^e) or (A_1^o, A_2^o) to correspond to even or odd solutions, since ϕ shares the symmetry of Φ for symmetric profiles.

5.1. $\lim_{V\to\infty} f_0^2 = \textit{constant} \neq 0$ *or 1*

This condition describes the physical configuration of separated caustics not close to a boundary. In this case, near a boundary ϕ is described by the exponential functions of § 3.1. The boundary condition (2.9) at $x = 1$ is applied to the representation (4.1) with the coefficients defined by (4.21). With some rearrangement the resulting equations can be put in the form

$$\tan\left(\frac{\nu\pi}{2}\right) = \frac{1}{2}\left(\frac{W'-V}{W'+V}\right) \mathrm{e}^{-2V\zeta} \tag{5.1a}$$

for ϕ^e and

$$\cot\left(\frac{\nu\pi}{2}\right) = -\frac{1}{2}\left(\frac{W'-V}{W'+V}\right) \mathrm{e}^{-2V\zeta} \tag{5.1b}$$

for ϕ^o, where

$$W' = \left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{-1}\left[W + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}x}\right)^{-1/2}\right] \tag{5.2}$$

evaluated at $x = 1$, with $\mathrm{d}\zeta/\mathrm{d}x$ a solution of (3.6b), and ζ evaluated at $x = 1$. Both (5.1a) and (5.1b) can be inverted in the form

$$\nu = q + 2\theta/\pi \tag{5.3}$$

where q is a non-negative integer and

$$\theta = \tan^{-1}\left[\frac{1}{2}\left(\frac{W'-V}{W'+V}\right) \mathrm{e}^{-2V\zeta}\right] \qquad x = 1. \tag{5.4}$$

Adding $\frac{1}{2}$ to both sides of (5.3) and using (4.13) gives

$$-\frac{V}{2\pi\mathrm{i}}\oint_{\Gamma_0} w\,\mathrm{d}x = q + \frac{1}{2} + \frac{2\theta}{\pi} \tag{5.5}$$

with

$$w = \mathrm{d}\zeta/\mathrm{d}x. \tag{5.6}$$

Equation (5.5) is the required eigenvalue equation. It is exact in the sense that both sides can be computed to arbitrary order in V^{-1} if a value for f_0^2 is given.

5.2. $\lim_{V\to\infty} f_0^2 = 1$

This condition describes a configuration where a caustic is close to a boundary. A similar calculation to § 5.1 above can be carried out using the Airy function representations (4.61) for ϕ^e and ϕ^o. The result is similar to (5.5):

$$-\frac{V}{2\pi\mathrm{i}}\oint_{\Gamma_0} w\,\mathrm{d}x = q + \frac{1}{2} + \frac{2\theta}{\pi} \tag{5.7}$$

where q is a non-negative integer and

$$\theta = \tan^{-1}\left(\frac{\mathrm{Ai}'(V^{2/3}\tau) + V^{-2/3}W''\,\mathrm{Ai}(V^{2/3}\tau)}{\mathrm{Bi}'(V^{2/3}\tau) + V^{-2/3}W''\,\mathrm{Bi}(V^{2/3}\tau)}\right) \tag{5.8}$$

$$W'' = \left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{-1}\left[W + \left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\tau}{\mathrm{d}x}\right)^{-1/2}\right] \tag{5.9}$$

and τ is a solution of (3.22b) evaluated, along with its derivatives, at $x = 1$. The prime on the Airy functions denotes differentiation with respect to argument.

5.3. $\lim_{V\to\infty} f_0^2 = 0$

This condition corresponds to degenerate caustics at the origin. In this case, boundary conditions are applied to the full uniform solution (3.30a) with $\Phi = \Phi^{\mathrm{e}}$ or Φ^{o} for even or odd solutions respectively. Using (3.35), applying the boundary condition (2.9) and rearranging results in

$$-\frac{V}{2\pi\mathrm{i}}\oint_{\Gamma_0} w\,\mathrm{d}x = q + \frac{1}{2} + \frac{2\theta}{\pi} \tag{5.10}$$

where q is a non-negative integer,

$$\theta = \tan^{-1}\left(\frac{\Phi_1^{\mathrm{e}'} + W'''\Phi_1^{\mathrm{e}}}{\Phi_2^{\mathrm{e}'} + W'''\Phi_2^{\mathrm{e}}}\right) \qquad (\text{q even}) \tag{5.11a}$$

$$= \tan^{-1}\left(\frac{\Phi_1^{\mathrm{o}'} + W'''\Phi_1^{\mathrm{o}}}{\Phi_2^{\mathrm{o}'} + W'''\Phi_2^{\mathrm{o}}}\right) \qquad (q \text{ odd}) \tag{5.11b}$$

and

$$W''' = \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1}\left[W + \left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\xi}{\mathrm{d}x}\right)^{-1/2}\right] \tag{5.12}$$

evaluated at $x = 1$. The prime on the Φ functions denotes differentiation with respect to ξ, and all x-dependent terms in (5.11) are evaluated at $x = 1$. The function ξ is a solution of (3.30b), and because of the smallness of f_0^2 the approximate solution (3.33) may be used.

6. Conclusion

This paper has described the asymptotic solution of a differential equation with boundary conditions at the ends of a finite interval which occurs in the theory of planar inhomogeneous waveguides, using a transformation on the independent variable to map the dependent variable onto a known solution of a canonical differential equation. This method is, in principle, capable of generating approximations to all asymptotic orders in the large parameter V, and we have considered a variety of representations valid over various parts of the domain of the differential equation, including a fully uniform representation valid over the whole of the domain.

No attempt has been made to solve the eigenvalue equations arising in this paper, for reasons of length; this problem is to be considered in the following paper.

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Appendix

Here it is desired to make the group properties referred to in § 3 more apparent. This Appendix is not intended to be rigorous, merely descriptive.

Consider the set of all differential equations of the form

$$\mathrm{d}^2\Phi/\mathrm{d}t^2 - V^2G^2\Phi = 0 \tag{A1}$$

where G is a member of a suitable class of functions; in our case, we take G^2 to be an analytic function of t. Let X denote the equation (A1). The transformation

$$G_2^2\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right)^2 = G_1^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}\left(\frac{\mathrm{d}t_2}{\mathrm{d}t_1}\right)^{-1/2} \tag{A2a}$$

$$\Phi_1 = (\mathrm{d}t_2/\mathrm{d}t_1)^{-1/2}\Phi_2 \tag{A2b}$$

transforms from

$$X_1\colon \mathrm{d}^2\Phi_1/\mathrm{d}t_1^2 - V^2G_1^2\Phi_1 = 0 \tag{A3}$$

to

$$X_2\colon \mathrm{d}^2\Phi_2/\mathrm{d}t_2^2 - V^2G_2^2\Phi_2 = 0. \tag{A4}$$

The transformation $X_1 \to X_2$ may be written symbolically as

$$X_2 = T_{21}X_1. \tag{A5}$$

Consider now an arbitrary finite set of functions $\{G_i\colon i \leq N\}$. This set generates $(N-1)^2$ transformations between different elements of $\{X_i\colon i \leq N\}$, and the set of these transformations is called T_0. The union of T_0 with the identity I (no transformation at all) is called the *transformation set*, T:

$$T = T_0 \cup I \tag{A6a}$$

$$T_0 = \{T_{ij}\colon i \leq N, j \leq N, i \neq j\}. \tag{A6b}$$

A multiplication operation (.) is defined on T as follows. If

$$X_i = T_{ij}X_j \qquad X_j = T_{jk}X_k \tag{A7a}$$

then

$$X_i = T_{ij}(T_{jk}X_k) = T'_{ik}X_k \tag{A7b}$$

and, by definition,

$$T_{ij} \, . \, T_{jk} = T'_{ik}. \tag{A8}$$

It will now be proved that $T'_{ik} \in T$. We have

$$G_j^2\left(\frac{\mathrm{d}t_j}{\mathrm{d}t_i}\right)^2 = G_i^2 - \frac{1}{V^2}\left(\frac{\mathrm{d}t_j}{\mathrm{d}t_i}\right)^{1/2}\frac{\mathrm{d}^2}{\mathrm{d}t_i^2}\left(\frac{\mathrm{d}t_j}{\mathrm{d}t_i}\right)^{-1/2} \tag{A9a}$$

$$G_k^2\left(\frac{dt_k}{dt_j}\right)^2 = G_j^2 - \frac{1}{V^2}\left(\frac{dt_k}{dt_j}\right)^{1/2}\frac{d^2}{dt_j^2}\left(\frac{dt_k}{dt_j}\right)^{-1/2}. \tag{A9b}$$

Multiplying (A9*b*) by $(dt_j/dt_i)^2$ and using (A9*a*) on the right leads to

$$G_k^2\left(\frac{dt_k}{dt_i}\right)^2 = G_i^2 - \frac{1}{V^2}\left(\frac{dt_k}{dt_i}\right)^{1/2}\frac{d^2}{dt_i^2}\left(\frac{dt_j}{dt_i}\right)^{-1/2} - \frac{1}{V^2}\left(\frac{dt_j}{dt_i}\right)^2\left(\frac{dt_k}{dt_j}\right)^{1/2}\frac{d^2}{dt_j^2}\left(\frac{dt_k}{dt_j}\right)^{-1/2} \tag{A10}$$

and it can be verified that the second and third terms on the right of (A10) can be simplified to give

$$G_k^2\left(\frac{dt_k}{dt_i}\right)^2 = G_i^2 - \frac{1}{V^2}\left(\frac{dt_k}{dt_i}\right)^{1/2}\frac{d^2}{dt_i^2}\left(\frac{dt_k}{dt_i}\right)^{-1/2}. \tag{A11a}$$

Furthermore,

$$\Phi_k = \left(\frac{dt_k}{dt_j}\right)^{-1/2}\Phi_j = \left(\frac{dt_k}{dt_j}\right)^{-1/2}\left(\frac{dt_j}{dt_i}\right)^{-1/2}\Phi_i$$

and so

$$\Phi_k = (dt_k/dt_i)^{-1/2}\Phi_i. \tag{A11b}$$

Equations (A11) define the transformation $T_{ik} \in T$. Therefore $T'_{ik} = T_{ik}$, and (A8) may be written

$$T_{ij} \,.\, T_{jk} = T_{ik}. \tag{A12}$$

Therefore the set T_0 is *closed under multiplication.*

The identity obviously belongs to T, and clearly

$$(T_{ij} \,.\, I)X_j = (I \,.\, T_{ij})X_j = T_{ij}X_j. \tag{A13}$$

Therefore the set T is *closed under multiplication and possesses an identity.*

By means very similar to the proof of closure, it can be shown that, if

$$T_{ij}X_j = X_i \tag{A14a}$$

then

$$X_j = T_{ji}X_i \tag{A14b}$$

i.e.

$$T_{ij} \,.\, T_{ji} = T_{ji} \,.\, T_{ij} = I \tag{A15}$$

defines an *inverse for every element of T which is also an element of T.*

Finally, the *associativity of multiplication on T* is demonstrated by means similar to the property of closure:

$$T_{ij} \,.\, (T_{jk} \,.\, T_{kl}) = (T_{ij} \,.\, T_{jk}) \,.\, T_{kl}. \tag{A16}$$

These properties define a group, the *transformation group.*

Imposing analyticity requirements on the functions $t_i(t_j)$ greatly restricts the admissible functions $\{G_i\}$.

The transformation set used in this paper is constructed using the following sets of variables:

$$t_1 = x \qquad G_1^2 = F^2 \tag{A17a}$$

$$t_2 = \zeta \qquad G_2^2 = 1 \tag{A17b}$$

$$t_3 = \eta \qquad G_3^2 = -1 \tag{A17c}$$

$$t_4 = \tau \qquad G_4^2 = \tau \tag{A17d}$$

$$t_5 = \xi \qquad G_5^2 = \xi^2 - \xi_0^2. \tag{A17e}$$

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